A BERNSTEIN RESULT AND COUNTEREXAMPLE FOR ENTIRE SOLUTIONS TO DONALDSON'S EQUATION

MICAH WARREN
UNIVERSITY OF OREGON

ABSTRACT. We show that convex entire solutions to Donaldson's equation are quadratic, using a result of Weiyong He. We also exhibit entire solutions to the Donaldson equation that are not of the form discussed by He. In the process we discover some non-trivial entire solutions to complex Monge-Ampère equations.

1. Introduction

In this note we show the following.

Theorem 1. *Suppose that u is a convex solution to the Donaldson equation on $\mathbb{R}\times\mathbb{R}^{n-1}=(t,x_2,...,x_n)$*

$$\tilde{\sigma}_2(D^2u)=u_{11}(u_{22}+u_{33}+...+u_{nn})-u_{12}^2-...-u_{1n}^2=1. \tag{1}$$

Then u is a quadratic function.

Donaldson introduced the operator

$$Q(D^2u)=u_{tt}\Delta u-|\nabla u_t|^2$$

arising in the study of the geometry of the space of volume forms on compact Riemannian manifolds [1]. On Euclidean space, (1) becomes an interesting non-symmetric fully nonlinear equation. Weiyong He has studied aspects of entire solutions on Euclidean space, and was able to show that [2, Theorem 2.1] if $u_{11}=const$, then the solution can be written in terms of solutions to Laplace equations.

Here we show that any convex solution must also satisfy $u_{11}=const$. It follows quickly that the solution must be quadratic. We also show that, in the absence of the convexity constraint, solutions exists for which $u_{11}=const$ fails.

Theorem 2. *There exists solutions to the Donaldson equation which are not of the form given by He.*

In real dimension 3 we note that solutions of (1) can be extended to solutions of the the complex Monge-Ampère equation on $\mathbb{C}^2$

$$\det\left(\partial\bar{\partial}u\right)=1 \tag{2}$$

and we can conclude the following.

Corollary 3. *There exist a nonflat solution of the complex Monge-Ampère equation (2) on $\mathbb{C}^2$ whose potential depends on only three real variables.*

The author's work is supported in part by the NSF via DMS-1161498.

2. Proof of Theorem 1

Lemma 4. *Suppose that K_h is the sublevel set $u \leq h$ of a nonnegative solution to*

$$\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + ... + u_{nn}) - u_{12}^2 - ... - u_{1n}^2 = 1.$$

Then for all ellipsoids $E \subset K_h$ such that if $A : E \to B_1$ is affine diffeomorphism with

$$A = Mx + \vec{b},$$

we have

$$\tilde{\sigma}_2(M^2) \geq \frac{1}{4}\frac{1}{h^2}.$$

Proof. Consider the function v on $\mathbb{R}^n$ defined by

$$v(x) = h|A(x)|^2.$$

On the boundary of E, we have

$$v(x) = h \geq u.$$

We have

$$Dv = 2hM\left(Mx + \vec{b}\right)$$
$$D^2v = 2hM^2.$$

Thus

$$\tilde{\sigma}_2(D^2v) = 4h^2\tilde{\sigma}_2(M^2).$$

Now suppose that

$$\tilde{\sigma}_2(M^2) < \frac{1}{4h^2}.$$

Then

$$\tilde{\sigma}_2(D^2v) < 1,$$

so v is a supersolution to the equation, and must lie strictly above the solution u. But v must vanish at $A^{-1}(0)$. Because u is nonnegative, this is a contradiction of the strong maximum principle. □

Proposition 5. *Suppose that u is an entire convex solution to*

$$\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + ... + u_{nn}) - u_{12}^2 - ... - u_{1n}^2 = 1$$

Then

$$\lim_{t\to\infty} u_1(t, 0, ..0) = \infty.$$

Proof. Assume not. Instead assume that $u_1 \leq A$. Assume that $u(0) = 0$ and $Du(0) = 0$, adjusting A if necessary. Then

$$u(t, 0, ..., 0) = \int_0^t u_1(s)ds \leq \int_0^t Ads \leq At.$$

Now consider the convex sublevel set $u \leq h$. This must contain the point

$$(\frac{h}{A}, 0, .., 0).$$

The level set $u = h$ intersect the other axes at

$$\begin{aligned}&(0, a_2(h), 0, ...,)\\&(0, 0, a_3(h), ..., 0)\\&\text{etc.}\end{aligned}$$

This level set is convex. It must contain the simplex with the above points as vertices, and this simplex must contain an ellipsoid E which has an affine transformation to the unit ball of the following form

$$A = Mx + \vec{b}$$

$$M = c_n \begin{pmatrix} \frac{A}{h} & & & \\ & \frac{1}{a_2} & & \\ & & \frac{1}{a_3} & \\ & & & ... \end{pmatrix}.$$

Thus

$$M^2 = c_n^2 \begin{pmatrix} \left(\frac{A}{h}\right)^2 & & & \\ & \left(\frac{1}{a_2}\right)^2 & & \\ & & \left(\frac{1}{a_3^2}\right)^2 & \\ & & & ... \end{pmatrix}$$

and

$$\tilde{\sigma}_2(M^2) = c_n^2 \left(\frac{A}{h}\right)^2 \left(\left(\frac{1}{a_2}\right)^2 + ... + \left(\frac{1}{a_n}\right)^2\right) \geq \frac{1}{4}\frac{1}{h^2}$$

with the latter inequality following from the previous lemma.

Thus

$$\left(\frac{1}{a_2}\right)^2 + \left(\frac{1}{a_3}\right)^2 + ... + \left(\frac{1}{a_n}\right)^2 \geq \frac{1}{c_n^2 4A^2}.$$

It follows that for some i,

$$\frac{1}{a_i^2} \geq \frac{1}{4\,(n-1)\,c_n^2 A^2}.$$

That is

$$a_i \leq 2\sqrt{n-1}c_n A.$$

Now to finish the argument, let

$$R = 2\sqrt{n-1}c_n A.$$

On a ball of radius R, there is some bound on the function (not a priori but depending on u) say $\bar{U}$. That is

$$u(x) \leq \bar{U} \text{ on } B_{R.}$$

Now by convexity for any large enough h the level set $u = h$ is non-empty and convex. Choose $h > \bar{U}$. According to the above argument, this level set must intersect some axis at a point less than R from the origin, which is a contradiction.

$\square$

Now using this Proposition, we may repeat the argument of He [2, section 3]: Letting $z = u_1(t,x)$ the map

$$\Phi : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$$
$$\Phi(t,x) = (z,x)$$

is a diffeomorphism. Thus for x fixed, there exists a unique $t = t(z,x)$ such that $z = u_1(t,x)$. Defining

$$\theta(z,x) = t(z,x)$$

the computations in [2, section 3] yield that θ is a harmonic function. It follows that $\frac{\partial \theta}{\partial z} = 1/u_{11}$ is a positive harmonic function, so must be constant. Now we have

$$u(t,x) = at^2 + tb(x) + g(x)$$

which satisfies [2, section 2]

$$\Delta b = 0$$
$$\Delta g = \frac{1}{2a}\left(1 + |\nabla b|^2\right).$$

Letting $t = 0$ we conclude that g is convex. Letting $t \to \pm\infty$ we conclude that b is convex and concave, so must be linear. It follows that $|\nabla b|$ is constant, and

$$\Delta g - c\,|x|^2$$

is a semi-convex harmonic function, which must be a quadratic.

3. Counterexamples

We use the method described in [3] and restrict to $n = 3$. Consider

$$u(t,x) = r^2 e^t + h(t)$$

where $r = \left(x_2^2 + x_3^2\right)^{1/2}$. At any point we may rotate $\mathbb{R}^2$ so that $x_2 = r$ and get

$$D^2 u = \begin{pmatrix} r^2 e^t + h''(t) & 2re^t & 0 \\ 2re^t & 2e^t & 0 \\ 0 & 0 & 2e^t \end{pmatrix}.$$

We compute

$$\tilde{\sigma}_2\left(D^2 u\right) = 4e^t\left(r^2 e^t + h''(t)\right) - 4r^2 e^{2t} = 4e^t h''(t).$$

Then

$$u = r^2 e^t + \frac{1}{4} e^{-t}$$

is a solution.

Now defining complex variables

$$z_1 = t + is$$
$$z_2 = x + iy$$

we can consider the function

$$u = \left(x^2 + y^2\right) e^t + \frac{1}{4} e^{-t}. \tag{3}$$

The function 3 satisfies the equation complex Monge-Ampère equation

$$\left(\partial_{z_1}\partial_{z_{\bar{1}}} u\right)\left(\partial_{z_2}\partial_{z_{\bar{2}}} u\right) - \left(\partial_{z_1}\partial_{z_{\bar{2}}} u\right)\left(\partial_{z_2}\partial_{z_{\bar{1}}} u\right) = 1.$$

One can check that the induced Ricci-flat complex metric

$$g_{i\bar{j}} = \partial_{z_i}\partial_{z_{\bar{j}}} u$$

on $\mathbb{C}^2$ is neither complete complete nor flat.

References

[1] Simon K. Donaldson. Nahm's equations and free-boundary problems. In *The many facets of geometry*, pages 71–91. Oxford Univ. Press, Oxford, 2010.
[2] Weiyong He. Entire solutions of Donaldson's equation. *Pacific J. Math.*, 256(2):359–363, 2012.
[3] Micah Warren. Non-polynomial entire solutions to σ_k equations. *arXiv:1214118*.

Fenton Hall, University of Oregon, Eugene, OR 97403
E-mail address: micahw@uoregon.edu